Solutions to short-answer questions

1 a Let the 3 consecutive integers be $n, n+1$ and $n+2$. Then,

$$\begin{aligned} n+(n+1)+(n+2) &= 3n+3 \\ &= 3(n+1) \end{aligned}$$

is divisible by 3.

b This statement is not true. For example, $1+2+3+4=10$ is not divisible by 4

2 (Method 1) If n is even then $n=2k$, for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} n^2-3n+1 &= (2k)^2-2(2k)+1 \\ &= 4k^2-4k+1 \\ &= 2(2k^2-2k)+1 \end{aligned}$$

is odd.
(Method 2) If n is even then n^2-3n+1 is of the form

$$\text{even} - \text{even} + \text{odd} = \text{odd}.$$

3 a (Contrapositive) If n is not even, then n^3 is not even. (Alternative) If n is odd, then n^3 is odd.

b If n is odd then $n=2k+1$, for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} n^3 &= (2k+1)^3 \\ &= 8k^3+12k^2+6k+1 \\ &= 2(4k^3+6k^2+3k)+1 \end{aligned}$$

is odd.

c This will be a proof by contradiction. Suppose $\sqrt[3]{6}$ is rational so that $\sqrt[3]{6}=\dfrac{p}{q}$ where $p, q \in \mathbb{Z}$. We can assume that p and q have no common factors (or else they could be cancelled). Then,

$$\begin{aligned} & p^3 = 6q^3 \qquad (1) \\ \Rightarrow\quad & p^3 \text{ is divisible by } 2 \\ \Rightarrow\quad & p \text{ is divisible by } 2 \\ \Rightarrow\quad & p = 2k \text{ for some } k \in \mathbb{N} \\ \Rightarrow\quad & (2k)^3 = 6q^3 (\text{substitituting into } (1)) \\ \Rightarrow\quad & 8k^3 = 6q^3 \\ \Rightarrow\quad & 4k^2 = 3q^2 \\ \Rightarrow\quad & q^2 \text{ is divisible by } 2 \\ \Rightarrow\quad & q \text{ is divisible by } 2. \end{aligned}$$

So p and q are both divisible by 2, which contradicts the fact that they have no factors in common.

4 a Suppose n is the first of three consecutive numbers. If n is divisible by 3 then there is nothing to prove. Otherwise, it is of the form $n=3k+1$ or $n=3k+2$. In the first case,

$$\begin{aligned} n &= 3k+1 \\ n+1 &= 3k+2 \\ n+2 &= 3k+3 = 3(k+1) \end{aligned}$$

so that the third number is divisible by 3. In the second case,

$$\begin{aligned} n &= 3k+2 \\ n+1 &= 3k+3 = 3(k+1) \\ n+2 &= 3k+4 \end{aligned}$$

so that the second number is divisible by 3.

b The expression can be readily factorised so that

$$\begin{aligned} n^3 + 3n^2 + 2n &= n(n^2 + 3n + 2) \\ &= n(n+1)(n+2) \end{aligned}$$

is the product of 3 consecutive integers. As one of these integers must be divisible by 3, the product must also be divisible by 3.

5 a if m and n are divisible by d then $m = pd$ and $n = qd$ for some $p, q \in \mathbb{Z}$. Therefore,

$$\begin{aligned} m - n &= pd - qd \\ &= d(p - q) \end{aligned}$$

is divisible by d.

b Take any two consecutive numbers n and $n+1$. If d divides n and $n+1$ then d must divide $(n+1) - n = 1$. As the only integer that divides 1 is 1, the highest common factor must be 1, as required.

c We know that any factor of 1002 and 999 must also divide $1002 - 999 = 3$. As the only factors of 3 are 1 and 3, the highest common factor must be 3.

6 a If $x = 9$ and $y = 16$ then the left hand side equals

$$\sqrt{9 + 16} = \sqrt{25} = 5$$

while the right hand side equals

$$\sqrt{9} + \sqrt{16} = 3 + 4 = 7.$$

b

$$\begin{aligned} (\Rightarrow) \quad \sqrt{x+y} &= \sqrt{x} + \sqrt{y} \\ \Rightarrow x + y &= \left(\sqrt{x} + \sqrt{y}\right)^2 \\ \Rightarrow x + y &= x + \sqrt{xy} + y \\ \Rightarrow 0 &= sqrtxy \\ \Rightarrow xy &= 0 \\ \Rightarrow x &= 0 \text{ or } y = 0 \end{aligned}$$

$(\Leftarrow)$ Suppose that $x = 0$ or $y = 0$. We can assume that $x = 0$. Then

$$\begin{aligned} \sqrt{x+y} &= \sqrt{y+0} \\ &= \sqrt{y} \\ &= \sqrt{y} + \sqrt{0} \\ &= \sqrt{y} + \sqrt{x}, \end{aligned}$$

as required.

7 (Case 1) If n is even then the expression is of the form

$$\text{even} + \text{even} + \text{even} = \text{even}.$$

(Case 1) If n is odd then the expression is of the form

$$\text{odd} + \text{odd} + \text{even} = \text{even}.$$

8 a If $a = b = c = d = 1$ then the left hand side equals

$$\frac{1}{1} + \frac{1}{1} = 2$$

while the right hand side equals

$$\frac{1+1}{1+1} = 1.$$

b first note that if $\frac{c}{d} > \frac{a}{b}$ then $bc > ad$. Therefore,

$$\begin{aligned}\frac{a+c}{b+d} - \frac{a}{b} &= \frac{b(a+c)}{b(b+d)} - \frac{a(b+d)}{b(b+d)} \\ &= \frac{b(a+c) - a(b+d)}{b(b+d)} \\ &= \frac{ab + bc - ab - ad}{b(b+d)} \\ &= \frac{bc - ad}{b(b+d)} \\ &> 0\end{aligned}$$

since $bc > ad$. This implies that

$$\frac{a+c}{b+d} > \frac{a}{b}.$$

Similarly, we can show that

$$\frac{a+c}{b+d} < \frac{c}{d}.$$

9 a $\boxed{P(n)}$

$6^n + 4$ is divisible by 10

$\boxed{P(1)}$

If $n = 1$ then

$$6^1 + 4 = 10$$

is divisible by 10. Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$6^k + 4 = 10m \qquad (1)$$

for some $m \in \mathbb{Z}$.

$\boxed{P(k+1)}$

$$\begin{aligned}6^{k+1} + 4 &= 6 \times 6^k + 4 \\ &= 6 \times (10m - 4) + 4 \quad \text{(by (1))} \\ &= 60m - 24 + 4 \\ &= 60m - 20 \times 3^k \\ &= 10(6m - 2)\end{aligned}$$

is divisible by 10. Therefore $P(k+1)$ is true.
Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

b $\boxed{P(n)}$

$$1^2 + 3^2 + \cdots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

$\boxed{P(1)}$

If $n = 1$ then LHS $= 1^2 = 1$and

$$\text{RHS} = \frac{1(2 \times 1 - 1)(2 \times 1 + 1)}{3} = 1.$$

Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$1^2+3^2+\cdots+(2k-1)^2=\frac{k(2k-1)(2k+1)}{3}. \qquad (1)$$

$\boxed{P(k+1)}$

$$\begin{aligned}\text{LHS of } P(k+1) &= 1^2+3^2+\cdots+(2k-1)^2+(2k+1)^2\\ &= \frac{k(2k-1)(2k+1)}{3}+(2k+1)^2 \quad (\text{by } (1))\\ &= \frac{k(2k-1)(2k+1)}{3}+\frac{3(2k+1)^2}{3}\\ &= \frac{k(2k-1)(2k+1)+3(2k+1)^2}{3}\\ &= \frac{(2k+1)(k(2k-1)+3(2k+1))}{3}\\ &= \frac{(2k+1)(2k^2-k+6k+3)}{3}\\ &= \frac{(2k+1)(2k+3)(k+1)}{3}\\ &= \frac{(k+1)(2k+1)(2k+3)}{3}\\ &= \frac{(k+1)(2(k+1)-1)(2(k+1)+1)}{3}\\ &= \text{RHS of } P(k+1)\end{aligned}$$

Therefore $P(k+1)$ is true.
Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

Solutions to multiple-choice questions

1 **E** The expression $m-3n$ is of the form

$$\text{even} \; - \; \text{odd} \; = \; \text{odd}.$$

2 **E** If m is divisible by 6 and n is divisible by 15 then $m=6p$ and $n=15q$ for $p,q\in\mathbb{Z}$. Therefore,

$$m\times n=90pq$$
$$m+n=6p+15q=3(2p+5q)$$

From these two expressions, it should be clear that A,B,C and D are true, while E might be false. For example, if $m=6$ and $n=15$ then $m+n=21$ is not divisible by 15.

3 **C** We obtain the contrapositive by switching P and Q and negating both. Therefore, the contrapositive will be

$$\text{not } Q \Rightarrow \text{not } P.$$

4 **B** We obtain the converse by switching P and Q. Therefore, the converse will be

$$Q \Rightarrow P.$$

5 **C** If $m+n=mn$ then

$$n=mn-m$$
$$n=m(n-1)$$

This means that n is divisible by $n-1$, which is only possible if $n=2$ or $n=0$. If $n=0$, then $m=0$. If $n=2$, then $m=2$. Therefore there are only two solutions, $(0,0)$ and $(2,2)$.

6 **D** The only statement that is true for all real numbers a, b and c is D. Counterexamples can be found for each of the other expressions, as shown below.

A $\frac{1}{3} < \frac{1}{2}$

B $\frac{1}{2} > \frac{1}{-1}$

C $3 \times -1 < 2 \times -1$

E $1^2 < (-2)^2$

7 **D** As n is the product of 3 consecutive integers, one of which will be divisible by 3 and one of which will be divisible by 2. The product will be then be divisible by $1, 2, 3$ and 6. On the other hand, it won't necessarily be divisible by 5 since $2 \times 3 \times 4$ is not divisible by 5.

8 **C** Each of the statements is true except the third. In this instance, $1 + 3$ is even, although 1 and 3 are not even.

Solutions to extended-response questions

1 a The number of dots can be calculated two ways, either by addition,

$$(1+2+3+4)+(1+2+3+4)$$

or by multiplication,

$$4 \times 5.$$

Equating these two expressions gives,

$$\begin{aligned}(1+2+3+4)+(1+2+3+4) &= 4 \times 5\\ 2(1+2+3+4) &= 4 \times 5\\ 1+2+3+4 &= \frac{4 \times 5}{2}\end{aligned}$$

The argument obviously generalises to more dots, giving equation (1).

b We have,

$$\begin{aligned}1+2\cdots+99 &= \frac{99 \times 100}{2}\\ &= 99 \times 50,\end{aligned}$$

which is divisible by 99.

c Suppose that m is the first number, so that the n connective numbers are

$$m, m+1, \ldots, m+n-1.$$

Then,

$$\begin{aligned}m+(m+1)+(m+2)+\cdots+(m+n-1) &= n \times m + (1+2+\cdots(n-1))\\ &= nm + \frac{(n-1)n}{2}\\ &= n\left(m + \frac{n-1}{2}\right)\end{aligned}$$

Since n is odd, $n-1$ is even. This means that $\frac{n-1}{2}$ is an integer. Therefore, the term in brackets is an integer, which means the expression is divisible by n.

d Since

$$1+2+\cdots+n = \frac{n(n+1)}{2},$$

we need to prove the following statement:

$\boxed{P(n)}$

$1^3 + 2^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$

$\boxed{P(1)}$

If $n = 1$ then

$$\text{LHS} = 1^3 = 1$$

and

$$\text{RHS} = \frac{1^2(1+1)^2}{4} = 1.$$

Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$1^3 + 2^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}. \quad (1)$$

$\boxed{P(k+1)}$

$$\begin{aligned}
\text{LHS of } P(k+1) &= 1^3 + 2^3 + \cdots + k^3 + (k+1)^3 \\
&= \frac{k^2(k+1)^2}{4} + (k+1)^3 \quad (\text{by } (1)) \\
&= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4} \\
&= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\
&= \frac{(k+1)^2(k^2 + 4(k+1))}{4} \\
&= \frac{(k+1)^2(k^2 + 4k + 4)}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4} \\
&= \frac{(k+1)^2((k+1)+1)^2}{4} \\
&= \text{RHS of } P(k+1)
\end{aligned}$$

Therefore $P(k+1)$ is true.
Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

2 **a** The first number is divisible by 2, the second by 3, the third by 4 and so on. As each number has a factor greater than 1, each is a composite number. Therefore this is a sequence of 9 consecutive composite numbers.

b We consider the this sequence of 10 consecutive numbers,

$$11! + 2, 11! + 3, \ldots, 11! + 11.$$

The first number is divisible by 2, the second by 3 and so on. Therefore as each number has a factor greater than 1, each is a composite number.

3 **a** Since (a, b, c) is a Pythagorean triple, we know that $a^2 + b^2 = c^2$. Then (na, nb, nc) is also a Pythagorean triple since,

$$\begin{aligned}
(na)^2 + (nb)^2 &= n^2a^2 + n^2b^2 \\
&= n^2(a^2 + b^2) \\
&= n^2(c^2) \\
&= (nc)^2,
\end{aligned}$$

as required.

b Suppose that $(n, n+1, n+2)$ is a Pythagorean triple. Then

$$\begin{aligned} n^2 + (n+1)^2 &= (n+2)^2 \\ n^2 + n^2 + 2n + 1 &= n^2 + 4n + 4 \\ n^2 - 2n - 3 &= 0 \\ (n-3)(n+1) &= 0 \\ n &= 3, -1. \end{aligned}$$

However, since $n > 0$, we obtain only one solution, $n = 3$, which corresponds to the famous $(3, 4, 5)$ triangle.

c Suppose some triple (a, b, c) contained the number 1. Then clearly, 1 will be the smallest number. Therefore, we can suppose that

$$\begin{aligned} 1^2 + b^2 &= c^2 \\ c^2 - b^2 &= 1 \\ (c-b)(c+b) &= 1 \end{aligned}$$

Since the only divisor of 1 is 1, we must have

$$\begin{aligned} &c + b = 1 \\ &c - b = 1 \\ &\Rightarrow b = 0 \text{ and } c = 1. \end{aligned}$$

This is a contradiction, since b must be a positive integer. Now suppose some triple (a, b, c) contained the number 2. Then 2 will be smallest number. Therefore, we can suppose that

$$\begin{aligned} 2^2 + b^2 &= c^2 \\ c^2 - b^2 &= 4 \\ (c-b)(c+b) &= 4 \end{aligned}$$

Since the only divisors of 4 are $1, 2$ and 4, we must have

$$\begin{aligned} &c + b = 4 \\ &c - b = 1 \\ &\Rightarrow b = \frac{3}{2}, c = \frac{5}{2} \end{aligned}$$

or

$$\begin{aligned} &c + b = 2 \\ &c - b = 2 \\ &\Rightarrow b = 0, c = 2 \end{aligned}$$

In both instances, we have a contradiction since b must be a positive integer.

4 a (Case 1) If $a = 3k + 1$ then

$$\begin{aligned} a^2 &= (3k+1)^2 \\ &= 9k^2 + 6k + 1 \\ &= 3(3k^2 + 2k) + 1 \end{aligned}$$

leaves a remainder of 1 when divided by 3.
(Case 2) If $a = 3k + 2$ then

$$\begin{aligned} a^2 &= (3k+2)^2 \\ &= 9k^2 + 12k + 4 \\ &= 9k^2 + 12k + 3 + 1 \\ &= 3(3k^2 + 4k + 1) + 1 \end{aligned}$$

also leaves a remainder of 1 when divided by 3.

b Suppose by way of contradiction that neither a nor b are divisible by 3. Then using the previous question, each of a^2 and b^2 leave a remainder of 1 when divided by 3. Therefore $a^2 = 3k+1$ and $b^2 = 3m+1$, for some $k, m \in \mathbb{Z}$. Therefore,

$$\begin{aligned} c^2 &= a^2 + b^2 \\ &= 3k+1+3m+1 \\ &= 3(k+m)+2. \end{aligned}$$

This means that c^2 leaves a remainder of 2 when divided by 3, which is not possible.

5 a $\boxed{P(n)}$

$n^2 + n$ is divisible by 2, where $n \in \mathbb{Z}$.

$\boxed{P(1)}$

If $n = 1$ then $1^2 + 1 = 2$ is divisible by 2. Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$k^2 + k = 2m \qquad (1)$$

for some $m \in \mathbb{Z}$.

$\boxed{P(k+1)}$

Letting $n = k+1$ we have,

$$\begin{aligned} (k+1)^2 + (k+1) &= k^2 + 2k + 1 + k + 1 \\ &= k^2 + 3k + 2 \\ &= (k^2 + k) + (2k + 2) \\ &= 2m + 2(k+1) \qquad \text{(by (1))} \\ &= 2(m + k + 1) \end{aligned}$$

is divisible by 2. Therefore $P(k+1)$ is true.
Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

b Since

$$n^2 + n = n(n+1)$$

is the product of two consecutive integers, one of them must be even. Therefore the product will also be even.

c If n is odd, then $n = 2k+1$. Therefore

$$\begin{aligned} n^2 - 1 &= (2k+1)^2 - 1 \\ &= 4k^2 + 4k + 1 - 1 \\ &= 4k^2 + 4k \\ &= 4k(k+1) \\ &= 4 \times 2k \qquad \text{(since the product of consecutive integers is even)} \\ &= 8k \end{aligned}$$

as required.

6 a If n is divisible by 8, then $n = 8k$ for some $k \in \mathbb{Z}$. Therefore $n^2 = (8k)^2 = 64k^2 = 8(8k^2)$ is divisible by 8.

b (Converse) If n^2 is divisible by 8, then n is divisible by 8.

c The converse is not true. For example, $4^2 = 16$ is divisible by 8 however 4 is not divisible by 8.

7 a There are many possibilities. For example $3 + 97 = 100$ and $5 + 97 = 102$.

b Suppose 101 could be written as the sum of two prime numbers. Then one of these primes must be 2, since all other pairs of primes have an even sum. Therefore $101 = 2 + 99$, however 99 is not prime.

c There are many possibilities. For example, $7 + 11 + 83 = 101$.

d Consider any odd integer n greater than 5. Then $n - 3$ will be an even number greater than 2. If the Goldbach Conjecture is true, then $n - 3$ is the sum of two primes, say p and q. Then $n = 3 + p + q$, as required.

8 a We have,

$$\begin{aligned}\frac{1}{n-1} - \frac{1}{n} &= \frac{n}{n(n-1)} - \frac{n-1}{n(n-1)} \\ &= \frac{n-(n-1)}{n(n-1)} \\ &= \frac{n-n+1}{n(n-1)} \\ &= \frac{1}{n(n-1)}.\end{aligned}$$

b Using the identity developed in the previous question, we have,

$$\begin{aligned}\frac{1}{2\times 1} + \frac{1}{3\times 2} + \cdots + \frac{1}{n(n+1)} &= \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{n-2} - \frac{1}{n-1} + \frac{1}{n-1} - \frac{1}{n} \\ &= \frac{1}{1} - \frac{1}{n} \\ &= 1 - \frac{1}{n}\end{aligned}$$

as required.

c True when $n = 2$ since $\frac{1}{2\times 1} = 1 - \frac{1}{2}$

Assume true for $n = k$

$$\frac{1}{2\times 1} + \frac{1}{3\times 2} + \cdots + \frac{1}{k(k+1)} = 1 - \frac{1}{k}$$

For $n = k + 1$

$$\begin{aligned}\frac{1}{2\times 1} + \frac{1}{3\times 2} + \cdots + \frac{1}{k(k-1)} + \frac{1}{(k+1)(k)} &= 1 - \frac{1}{k} + \frac{1}{(k+1)(k)} \\ &= 1 - \frac{1}{k+1}\end{aligned}$$

d Since $k^2 > k(k-1)$ for all $k \in \mathbb{N}$,

$$\begin{aligned}\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \cdots + \frac{1}{n^2} &= \frac{1}{1^2} + \left(\frac{1}{2^2} + \frac{1}{3^2} \cdots + \frac{1}{n^2}\right) \\ &< \frac{1}{1^2} + \left(\frac{1}{2\times 1} + \frac{1}{3\times 2} \cdots + \frac{1}{n(n-1)}\right) \\ &= \frac{1}{1^2} + 1 - \frac{1}{n} \\ &= 2 - \frac{1}{n} \\ &< 2,\end{aligned}$$

as required.

9 a We have,

$$\begin{aligned}\frac{x+y}{2}-\sqrt{xy} &= \frac{a^2+b^2}{2}-\sqrt{a^2b^2}\\ &= \frac{a^2+b^2}{2}-ab\\ &= \frac{a^2+b^2}{2}-\frac{2ab}{2}\\ &= \frac{a^2-2ab+b^2}{2}\\ &= \frac{(a-b)^2}{2}\\ &\geq 0.\end{aligned}$$

It is also worth noting that we get equality if and only if $x=y$.

b i Using the above inequality, we obtain,

$$\begin{aligned}a+\frac{1}{a} &\geq 2\sqrt{a\cdot\frac{1}{a}}\\ &= 2\sqrt{1}\\ &= 2.\end{aligned}$$

as required.

ii Using the above inequality three times, we obtain,

$$\begin{aligned}(a+b)(b+c)(c+a) &\geq 2\sqrt{ab}\times 2\sqrt{bc}\times 2\sqrt{ca}\\ &= 8(\sqrt{a})^2(\sqrt{b})^2(\sqrt{c})^2\\ &= 8abc,\end{aligned}$$

as required.

iii This inequality is a little trickier. We have,

$$\begin{aligned}a^2+b^2+c^2 &= \left(\frac{a^2}{2}+\frac{b^2}{2}\right)+\left(\frac{b^2}{2}+\frac{c^2}{2}\right)+\left(\frac{a^2}{2}+\frac{c^2}{2}\right)\\ &= \frac{a^2+b^2}{2}+\frac{b^2+c^2}{2}+\frac{a^2+c^2}{2}\\ &\geq \sqrt{a^2b^2}+\sqrt{b^2c^2}+\sqrt{a^2c^2}\\ &= ab+bc+ac,\end{aligned}$$

as required.

c If a rectangle has length x and width y then its perimeter will be $2x+2y$. A square with the same perimeter will have side length,

$$\frac{2x+2y}{4}=\frac{x+y}{2}.$$

Therefore,

$$A(\text{square})=\left(\frac{x+y}{2}\right)^2\geq xy=A(\text{rectangle}).$$

10 We show that it is only possible for Kaye to be the liar.

case 1

Suppose Jaye is lying

$\Rightarrow$ Kaye is not lying

$\Rightarrow$ Elle is lying

$\Rightarrow$ There are two liars

$\Rightarrow$ This is impossible.

case 2

Suppose Kaye is lying

$\Rightarrow$ Jaye is not lying and Elle is not lying

$\Rightarrow$ Kaye is the only liar

case 3

Suppose Elle is lying

$\Rightarrow$ Mina is not lying

$\Rightarrow$ Karl is lying

$\Rightarrow$ There are two liars

$\Rightarrow$ This is impossible.

11 First note that the four sentences can be recast as:

- Exactly three of these statements are true.
- Exactly two of these statements are true.
- Exactly one of these statements are true.
- None of these statements are true.

At most one of these statements can be true, or else we obtain a contradiction. If none of the statements is true, then the last statement is true. This means that at least one of the statements is true. This also gives a contradiction. Therefore, only one of the statements is true, that is, the third statement.

12a There is only one possibility,

$\boxed{1,2,4,8}$ $\boxed{3,5,6,7}$

b We know that we can split the numbers $1, 2, \ldots, 8$,

$\boxed{1,2,4,8}$ $\boxed{3,5,6,7}$

Deleting the largest number, 8, will give a splitting of $1, 2, \ldots, 7$.

$\boxed{1,2,4}$ $\boxed{3,5,6,7}$

Continuing this process, deleting the 7, will be a splitting of the numbers $1, 2, \ldots, 6$, and so on.

c We first note that if a set can be split then two numbers can't appear in the same group as their difference. To see this, if x and y and $x - y$ all belong to the same group then $(x - y) + y = x$. Let's now try to split the numbers $1, 2, \ldots, 9$. Call the two groups X and Y. We can assume that $1 \in X$. We now consider four cases for the groups containing elements 2 and 9.

(case 1) Suppose $2 \in X$ and $9 \in X$

Reason	X	Y	Reason
(assumed)	1		
(assumed)	2		
(assumed)	9		
		3	$(1, 2 \in X)$
		7	$(2, 9 \in X)$
$(3, 7 \in Y)$	4		
		5	$(1, 4 \in X)$
		6	$(2, 4 \in X)$
$(5, 6 \in Y)$	8		

This doesn't work, since X is forced to contain the numbers $1, 8$ and 9.

(case 2) Suppose $2 \in X$ and $9 \in Y$

Reason	X	Y	Reason
(assumed)	1		
(assumed)	2		
		9	(assumed)
		3	$(1, 2 \in X)$
$(3, 9 \in Y)$	6		
		4	$(2, 6 \in X)$
		5	$(1, 6 \in X)$

This doesn't work, since Y is forced to contain the numbers $4, 5$ and 9.

(case 3) Suppose $2 \in Y$ and $9 \in X$

Reason	X	Y	Reason
(assumed)	1		
		2	(assumed)
(assumed)	9		
		8	$(1, 9 \in X)$
$(2, 8 \in Y)$	6		
		3	$(6, 8 \in X)$
$(2, 8 \in Y)$	5		$(3, 8 \in X)$

This doesn't work, since X is forced to contain the numbers $1, 5$ and 6.

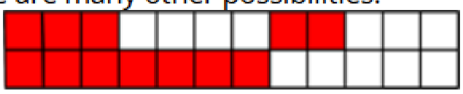
(case 4) Suppose $2 \in Y$ and $9 \in Y$

Reason	X	Y	Reason
(assumed)	1		
		2	(assumed)
		9	(assumed)
$(2, 9 \in Y)$	7		
		6	$(1, 7 \in X)$
$(2, 8 \in Y)$	4		
		3	$(4, 7 \in X)$

This doesn't work, since Y is forced to contain the numbers $3, 6$ and 9.

d If the numbers $1, 2, \ldots, n$ could be split, where $n \geq 9$, then we could successively eliminate the largest term to obtain a splitting of the numbers $1, 2, \ldots, 9$. However, we already know that this is impossible.

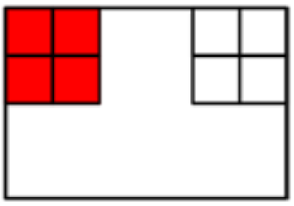
13a A suitable tiling is shown below. There are many other possibilities.



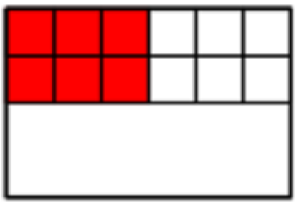
b Tile E must go into a corner. This is because there are only two other tiles (A and B) that it can go next to. Tile F must also go into a corner. This is because there are only two other tiles (B and C) that it can go next to.

(Case 1) Tile E and tile F are in different rows
Since tile B must go next to both tiles E and F, this is impossible.

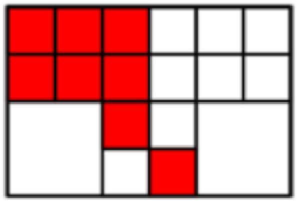
(Case 2) Tile E and tile F are in the same row
Assume tile F is in the top left position.
Then tile E goes in the top right position.



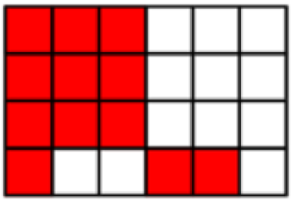
Therefore tile B must go between them.



Tile C must then go beneath tile F and tile A must go beneath tile E. Consequently, tile D must go beneath tile B. Therefore, there is only one valid orientation of tile D.



This fixes the orientation of tiles A and C.



Since tile F could have gone into any one of the four corners, there are only four ways to tile the grid.